

Counting functions for branched covers of elliptic curves and quasi-modular forms

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Abstract: We prove that each counting function of the m -simple branched covers with a fixed genus of an elliptic curve is expressed as a polynomial of the Eisenstein series E_2 , E_4 and E_6 . The special case $m = 2$ is considered by Dijkgraaf.

1 Introduction

We consider the counting function

$$F_g^{(m)}(q) = \sum_{d \geq 1} N_{g,d}^{(m)} q^d$$

of the branched covers of an elliptic curve. Here, $N_{g,d}^{(m)}$ is the (weighted) number of isomorphism classes of branched covers, with genus $g(> 1)$, degree d , and ramification index (m, m, \dots, m) , of an elliptic curve. Such a cover is called an m -simple cover. Our aim is to prove that the formal power series $F_g^{(m)}$ converges to a function belonging to the graded ring of *quasi*-modular forms with respect to the full modular group $SL(2, \mathbf{Z})$, and indeed that it can be expressed as a polynomial of the Eisenstein series E_2 , E_4 and E_6 with rational coefficients.

For $m = 2$, an m -simple branched cover is a simple branched cover. Dijkgraaf [3] has proved that the counting function $F_g^{(2)}(q)$ is a quasi-modular form with respect to $SL(2, \mathbf{Z})$. Our result is a generalization of this result for arbitrary $m \geq 2$.

The proof [3] for $m = 2$ employs the ‘Fermionic formula’ [5] of the partition function,

$$\begin{aligned} & \exp \left(\sum_{g=1}^{\infty} F_g^{(2)}(q) \frac{X^{2g-2}}{(2g-2)!} \right) \\ &= q^{-1/24} \operatorname{Res}_{z=0} \left(\prod_{p \in \frac{1}{2} + \mathbf{Z}_{\geq 0}} (1 + zq^p \exp(p^2 X/2))(1 + z^{-1}q^p \exp(-p^2 X/2)) \frac{dz}{z} \right), \end{aligned}$$

whose quasi-modularity was proven by Kaneko and Zagier [7]. The quasi-modularity of the counting function $F_g^{(2)}$ supports the mirror symmetry for an elliptic curve. For $m \geq 3$, although the relation between the counting function $F_g^{(m)}$ and the theory of mirror symmetry has not yet been clarified, the quasi-modularity of the counting function has been shown to hold.

The proof of our main theorem, Theorem 9, implies that all counting functions $F_q^{(m)}$ with $m \geq 2$ and $g > 1$ live in the infinite product

$$V(q, t_2, t_3, \dots) = \exp\left(-\sum_{j=1}^{\infty} \xi(-j)t_j\right) \times \operatorname{Res}_{z=0} \left(\prod_{p \in \frac{1}{2} + \mathbf{Z}_{\geq 0}} (1 + zq^p \exp(\sum_{k \geq 2} p^k t_k)) (1 + z^{-1}q^p \exp(-\sum_{k \geq 2} (-p)^k t_k)) \frac{dz}{z} \right),$$

with the infinite set of variables q, t_2, t_3, \dots , where the renormalizing factor $\xi(-j)$ is the special value of a Hurwitz zeta function. To be more precise, we show that every $F_g^{(m)}$ is a linear combination of the Taylor coefficients of the function V . Then, the quasi-modularity of the counting function $F_g^{(m)}$ is derived from the corresponding property for V , which was established by Bloch-Okounkov [2]. The key step in the proof is Proposition 4. In fact, the counting function is expressed as a combinatorial sum related to the symmetric group S_d . This expression depends strongly on the size d of the symmetric group S_d . To complete the summation in d , we need some formula free of d . Such a formula is obtained in Proposition 4.

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2 Counting functions

2.1 m -simple branched cover

We fix an elliptic curve E over \mathbf{C} and an integer $m \geq 2$. A pair (f, C) consisting of a (smooth complex) curve C and a holomorphic map $f : C \rightarrow E$ is an m -simple branched cover if the following three conditions are satisfied:

- (i) C is connected.
- (ii) For any $P \in C$, the branching index $e(P) = 1$ or m .
- (iii) If $P \neq P'$ and $e(P) = e(P') = m$, then $f(P) \neq f(P')$.

In the case $m = 2$, a 2-simple branched cover is usually called a ‘simple branched cover’. An m -simple branched cover is a natural generalization of a simple branched cover. If f is of degree d and the curve C is of genus g , then the pair (f, C) is said to be of genus g and degree d .

Two m -simple branched covers (f, C) and (f', C') are isomorphic if there is an isomorphism $\varphi : C \rightarrow C'$ such that $f = f' \circ \varphi$. The group of automorphisms on (f, C) is denoted $\text{Aut}(f, C)$ [or simply $\text{Aut}(f)$]. We will see that this is a finite group.

By the Riemann-Hurwitz formula (e.g., [6]), we have

$$2g(C) - 2 = d(2g(E) - 2) + \sum_{P \in C} (e(P) - 1).$$

Thus the number b of branch points and the genus g of the curve C always satisfy the relation $2g - 2 = (m - 1)b$. Note that the genus g does not depend on the degree d . This relation implies that the number b of branch points should be even if m is even. If m is odd, the number of branch points is arbitrary. The case $g = 1$ corresponds to the case $b = 0$; that is, the cover $f : C \rightarrow E$ is unramified.

We choose b (distinct) points $P_1, \dots, P_b \in E$. For $g = 1 + (m - 1)b/2$, let $X_{g,d} = X_{g,d}^{(m)}$ be the set of isomorphism classes of m -simple branched covers of genus g and degree d such that $e(P_i) = m$ for $i = 1, \dots, b$, and $e(P') = 1$ for $P' \in E \setminus \{P_1, \dots, P_b\}$. We will see that $X_{g,d}$ is a finite set and does not depend on the choice of the set of branch points P_1, \dots, P_b . In fact, $X_{g,d}$ can also be regarded as the fiber in the fibration

$$X_{g,d} \rightarrow \mathcal{M}_g(E, d) \rightarrow E_b,$$

where $\mathcal{M}_g(E, d)$ is the Hurwitz space of m -simple branched covers, and E_b is the configuration space of unordered b -points on E .

We count the (weighted) number of elements of $X_{g,d}$ so that

$$N_{g,d} = \sum_{f \in X_{g,d}} \frac{1}{|\text{Aut}(f)|}.$$

Note that $N_{g,d} = 0$ unless $2(g - 1) \in (m - 1)\mathbf{Z}_{\geq 0}$. The generating functions F_g for $g > 1$ are now defined by

$$F_g(q) = F_g^{(m)}(q) = \sum_{d \geq 1} N_{g,d} q^d.$$

These functions are called the ‘counting functions’.

It is necessary to define F_1 separately. The case $g = 1$ must be treated separately because covers in this case are unbranched ($b = 0$). Note that neither $X_{1,d}$ or $N_{1,d}$ depends on m . Then we employ the definition of $F_1(q)$ introduced for the case $m = 2$ [3, §2]:

$$F_1(q) = -\frac{1}{24} \log q + \sum_{d \geq 1} N_{1,d} q^d.$$

Here, the first term can be considered the contribution of the constant map (the map of degree zero) which is not a stable map. Since $N_{1,d} = \sigma_1(d)/d$, where $\sigma_1(d)$ is the sum of all divisors of d , we have the expression

$$F_1(q) = -\log \eta(q),$$

where we denote the Dedekind eta function by

$$\eta(q) = q^{1/24} \prod_{n \geq 1} (1 - q^n).$$

Next, we introduce a two-variable partition function Z ,

$$\begin{aligned} Z(q, X) = Z^{(m)}(q, X) &:= \exp \left(\sum_{g \geq 1} F_g(q) \frac{X^{(2g-2)/(m-1)}}{((2g-2)/(m-1))!} \right) \\ &= \exp \left(\sum_{b \geq 0} F_{1+(m-1)b/2}(q) \frac{X^b}{b!} \right), \end{aligned}$$

which is a formal power series in q . We see that

$$\eta(q)Z(q, X) = \exp \left(\sum_{g \geq 2} F_g(q) \frac{X^{(2g-2)/(m-1)}}{((2g-2)/(m-1))!} \right) \quad (1)$$

$$= \exp \left(\sum_{b \geq 1} F_{1+(m-1)b/2}(q) \frac{X^b}{b!} \right). \quad (2)$$

In the definition of the counting function F_g , we consider only connected covers. We also introduce the partition function \hat{Z} of the counting functions of covers, which are not necessarily connected. Let $\hat{X}_{g,d}$ be the set of isomorphism classes of m -simple branched covers, which are not necessarily connected, of genus g and degree d . In other words, for $\hat{X}_{g,d}$, we impose conditions (ii) and (iii), but we drop condition (i). We define the corresponding (weighted) number of elements of $\hat{X}_{g,d}$ by

$$\hat{N}_{g,d} = \sum_{f \in \hat{X}_{g,d}} \frac{1}{|\text{Aut}(f)|},$$

the modified counting function \hat{F}_g for $g \geq 1$ by

$$\hat{F}_g(q) = \sum_{d \geq 1} \hat{N}_{g,d} q^d,$$

and its generating function \hat{Z} by

$$\begin{aligned} \hat{Z}(q, X) &= \sum_{g \geq 1} \hat{F}_g(q) \frac{X^{(2g-2)/(m-1)}}{((2g-2)/(m-1))!} \\ &= \sum_{b \geq 0} \hat{F}_{1+(m-1)b/2}(q) \frac{X^b}{b!}. \end{aligned}$$

The relation between these two functions is given as follows.

Lemma 1 *We have the relation $\hat{Z}(q, X) = q^{1/24} Z(q, X)$.*

Proof: This follows from a standard argument [3]. □

2.2 Representations of the fundamental group

The weighted number $\hat{N}_{g,d}$ of covers which are not necessarily connected is expressed in terms of representations of the fundamental group of the punctured elliptic curve.

Let π_1^b be the fundamental group of the b -punctured curve $E \setminus \{P_1, \dots, P_b\}$. It is known that the group π_1^b can be expressed in terms of the generators and relations as

$$\pi_1^b = \langle \alpha, \beta, \gamma_1, \dots, \gamma_b \mid \gamma_1 \cdots \gamma_b = \alpha \beta \alpha^{-1} \beta^{-1} \rangle.$$

Here, we denote the simple curve around a point P_i by $\gamma_i \in \pi_1(E')$.

Let S_d be the symmetric group S_d on d elements, and let $c^{(m)}$ be the conjugacy class of S_d of type $(m, 1^{d-m})$. In other words, the class $c^{(m)}$ consists of cycles of length m . We define

$$\Phi_{g,d} = \Phi_{g,d}^{(m)} = \{\varphi \in \text{Hom}(\pi_1^b, S_d) \mid \varphi(\gamma_i) \in c^{(m)} \text{ for } i = 1, \dots, b\},$$

where “Hom” represents the set of group homomorphisms. The symmetric group S_d acts on $\Phi_{g,d}$ according to

$$\varphi^\sigma(\gamma) = \sigma^{-1} \varphi(\gamma) \sigma, \quad \sigma \in S_d, \varphi \in \Phi_{g,d}.$$

Lemma 2 (i) *As a set, we have the bijection $\hat{X}_{g,d} \cong \Phi_{g,d}/S_d$.*

$$(ii) \hat{N}_{g,d} = |\Phi_{g,d}|/|S_d|.$$

Proof: (i) Let $E' = E \setminus \{P_1, \dots, P_b\}$ be a punctured curve. Let us choose a base point $P_0 \in E'$ as a base point. Then the fundamental group $\pi_1(E') = \pi_1(E', P_0)$ is isomorphic to π_1^b . For an $f \in \hat{X}_{g,d}$, we construct the corresponding map $\varphi \in \Phi_{g,d}$. Let $f^{-1}(P_0) = \{Q_1, \dots, Q_d\}$. Then we have the natural map

$$\varphi : \pi_1^b \cong \pi_1(E') \rightarrow \text{Aut}(f^{-1}(P_0)) \cong S_d.$$

Conversely, for each $\varphi \in \Phi_{g,d}$, we construct a covering $f \in \hat{X}_{g,d}$. We denote the universal covering of E' by E'^{univ} . Let $C' = E'^{univ} \times_{\varphi} \{1, \dots, d\} = E' \times \{1, \dots, d\} / \sim$, where $(x, i) \sim (\gamma x, \varphi(\gamma)i)$ when $\gamma \in \pi_1(E')$, $x \in E'^{univ}$ and $1 \leq i \leq d$. Then the natural projection $f' : C' \rightarrow E'^{univ} / \pi_1^b = E'$ is a covering of degree d . This extends to a ramified covering $f : C \rightarrow E$. It is easy to see that this construction gives the required bijection.

(ii) Under the bijection in (i), the group $\text{Aut}(f)$ of automorphisms corresponds to the stabilizer subgroup of S_d at φ . This implies that

$$|\text{Aut}(f)| = \#\{\sigma \in S_d \mid \varphi = \varphi^\sigma\}.$$

Then we have

$$\hat{N}_{g,d} = \sum_f \frac{1}{|\text{Aut}(f)|} = \frac{1}{|S_d|} \sum_f \#\{\varphi^\sigma \mid \sigma \in S_d, \varphi \text{ corresponds to } f\} = |\Phi_{g,d}|/|S_d|.$$

□

2.3 Irreducible characters of symmetric group

The number of group homomorphisms appearing in the previous lemma is written as a sum over the irreducible representations of the permutation group.

A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ of d is a non-increasing sequence $\lambda_1 \geq \lambda_2 \geq \dots$ of non-negative integers such that $\sum_{i=1}^d \lambda_i = d$. We denote the set of all partitions of d by \mathcal{P}_d . It is known that the set of irreducible representations of the symmetric group S_d is parametrized by \mathcal{P}_d . For each $\lambda \in \mathcal{P}_d$, we denote the corresponding irreducible character by χ_λ . Since a character is a class function, the value $\chi_\lambda(c)$ is well-defined for each conjugacy class c of S_d . We introduce the modified character

$$f_\lambda(c) = \frac{|c| \cdot \chi_\lambda(c)}{\dim \lambda},$$

where $|c|$ is the number of elements in the conjugacy class c , and $\dim \lambda$ is the dimension of the representation λ , that is, the value of $\chi_\lambda(e)$ at the identity of S_d .

Lemma 3 *For $g = 1 + (m-1)b/2$, we have*

$$|\Phi_{g,d}^{(m)}|/|S_d| = \sum_{\lambda \in \mathcal{P}_d} f_\lambda(c^{(m)})^b.$$

Proof : We apply the formula in Lemma 4 of [3] with $G = S_d$, $R = \mathcal{P}_d$, $c_1 = \dots = c_N = c^{(m)}$, $h = 1$ and $N = b$. □

2.4 Frobenius notation

Now we recall properties of Frobenius coordinates of partitions and shifted symmetric functions. Our Frobenius coordinates are parametrized by half-integers, not by integers, as is explained below.

For a partition $\lambda = (\lambda_1, \dots, \lambda_d) \in \mathcal{P}_d$, we define the shifted partition $\tilde{\lambda} = (\tilde{\lambda}_1, \dots, \tilde{\lambda}_d)$ by $\tilde{\lambda}_i = \lambda_i - i + \frac{1}{2}$. Let I be the set of positive half-integers, $I = \frac{1}{2} + \mathbf{Z}_{\geq 0} = \{\frac{1}{2}, \frac{3}{2}, \dots\}$. A partition λ gives us two subsets $P, Q \subset I$ such that

$$\begin{aligned} P &= \{\tilde{\lambda}_i \mid \tilde{\lambda}_i > 0, i = 1, \dots, d\}, \\ Q &= \{1/2, 3/2, \dots, (2d-1)/2\} \setminus \{-\tilde{\lambda}_i \mid -\tilde{\lambda}_i > 0, i = 1, \dots, d\} = \{\tilde{\lambda}'_i \mid \tilde{\lambda}'_i, i = 1, \dots, d\}, \end{aligned}$$

where λ' is the conjugate partition of λ . Then the cardinality of P equals that of Q . Conversely, for a given pair of subsets $P, Q \subset I$ with $|P| = |Q|$, we have the corresponding the partition $\lambda \in \mathcal{P}_d$ with $d = |P| + |Q| = 2|P|$.

We remark that our Frobenius coordinates (P, Q) are shifted by $1/2$ from the Frobenius coordinates $(\alpha_1, \dots, \alpha_r \mid \beta_1, \dots, \beta_r)$ introduced in Section I.1 of [8]. The precise relation is

$$P = \{\alpha_1 + \frac{1}{2}, \alpha_2 + \frac{1}{2}, \dots, \alpha_r + \frac{1}{2}\}, \quad Q = \{\beta_1 + \frac{1}{2}, \beta_2 + \frac{1}{2}, \dots, \beta_r + \frac{1}{2}\}.$$

For $k \in \mathbf{Z}_{\geq 0}$ we define

$$\tilde{p}_k(\lambda) = \sum_{i=1}^d \left(\tilde{\lambda}_i^k - (-i + \frac{1}{2})^k \right).$$

(This function is written as $p_k(\lambda)$ in (5.4) of [2]. For example, $\tilde{p}_0(\lambda) = 0$, $\tilde{p}_1(\lambda) = d$. From I.1.4 of [8] we have the relation

$$\sum_{i=1}^d (t^{\tilde{\lambda}_i} - t^{-i+\frac{1}{2}}) = \sum_{p \in P} t^p - \sum_{p \in Q} t^{-p},$$

where (P, Q) is the Frobenius coordinates of the partition λ . As a corollary, we have [2, (5.4)]

$$\tilde{p}_k(\lambda) = \sum_{i=1}^d \left(\tilde{\lambda}_i^k - (-i + \frac{1}{2})^k \right) = \sum_{p \in P} p^k - \sum_{p \in Q} (-p)^k.$$

This is a power-sum symmetric functions in $\tilde{\lambda} = (\tilde{\lambda}_1, \dots, \tilde{\lambda}_d)$ plus some polynomial in d of degree $k + 1$. We now introduce two additional polynomials symmetric in the $\tilde{\lambda}_i$. Let $e_j(\tilde{\lambda})$ be the j th elementary symmetric function and $h_j(\tilde{\lambda})$ the j th complete symmetric function, defined by

$$\begin{aligned} e_j(\tilde{\lambda}) &= \sum_{1 \leq i_1 < \dots < i_j \leq d} \tilde{\lambda}_{i_1} \cdots \tilde{\lambda}_{i_j}, \\ h_j(\tilde{\lambda}) &= \sum_{1 \leq i_1 \leq \dots \leq i_j \leq d} \tilde{\lambda}_{i_1} \cdots \tilde{\lambda}_{i_j}. \end{aligned}$$

These two functions can be expressed as polynomials in power-sum symmetric functions, and thus as polynomials in $\tilde{p}_k(\lambda)$ and d .

2.5 Character formula

The character value $f_\lambda(c^{(m)})$ can be written in terms of $\tilde{p}_k(\lambda)$. Although the character depends strongly on the rank d of the symmetric group S_d , the following expression is independent of d . It is thus useful for summation over d .

Proposition 4 *There exists a polynomial $\phi_m(Y_1, \dots, Y_m) \in \mathbf{Q}[Y_1, \dots, Y_m]$ such that for all $d \geq 1$ and $\lambda \in \mathcal{P}_d$, we have*

$$f_\lambda(c^{(m)}) = \phi_m(\tilde{p}_1(\lambda), \dots, \tilde{p}_m(\lambda)).$$

Proof: We consider a partition $\lambda = (\lambda_1, \dots, \lambda_d)$. Let

$$\mu_i = \lambda_i + d - i = \tilde{\lambda}_i + d - \frac{1}{2},$$

and $\varphi(x) = \prod_{i=1}^d (x - \mu_i)$. Then, from Example I.7.7 in [8], we have

$$f_\lambda(c^{(m)}) = \frac{1}{m^2} \operatorname{Res}_{x=\infty} \left(\frac{x(x-1) \cdots (x-m+1) \varphi(x-m)}{\varphi(x)} dx \right),$$

where the expression “Res” denotes the residue. Since $\varphi(x + d - \frac{1}{2}) = \prod_{i=1}^d (x - \tilde{\lambda}_i)$, we obtain

$$\begin{aligned} & f_{\lambda}(c^{(m)}) \\ &= \frac{1}{m^2} \operatorname{Res}_{x=\infty} \left((x + d - \frac{1}{2})(x + d - \frac{3}{2}) \cdots (x + d - m + \frac{1}{2}) \frac{\varphi(x - m + d - \frac{1}{2})}{\varphi(x + d - \frac{1}{2})} dx \right) \\ &= -\frac{1}{m^2} \operatorname{Res}_{y=0} \left((1 + (d - \frac{1}{2})y)(1 + (d - \frac{3}{2})y) \cdots (1 + (d - m + \frac{1}{2})y) \frac{\prod_{i=1}^d (1 - (m + \tilde{\lambda}_i)y)}{\prod_{i=1}^d (1 - \tilde{\lambda}_i y)} \frac{dy}{y^{m+2}} \right) \end{aligned}$$

by changing coordinates. The products appearing here are generating functions of elementary (resp. complete) symmetric functions:

$$\begin{aligned} \prod_{i=1}^d (1 - (m + \tilde{\lambda}_i)y) &= \sum_{j=0}^d (1 - my)^{d-j} (-y)^j e_j(\tilde{\lambda}), \\ \prod_{i=1}^d (1 - \tilde{\lambda}_i y)^{-1} &= \sum_{j=0}^{\infty} y^j h_j(\tilde{\lambda}), \end{aligned}$$

Then,

$$\begin{aligned} & f_{\lambda}(c^{(m)}) \\ &= -\frac{1}{m^2} \sum_{i=0}^d \sum_{j=0}^{\infty} e_i(\tilde{\lambda}) h_j(\tilde{\lambda}) \times \\ & \quad \operatorname{Res}_{y=0} \left((1 + (d - \frac{1}{2})y)(1 + (d - \frac{3}{2})y) \cdots (1 + (d - m + \frac{1}{2})y) (1 - my)^{d-i} (-y)^i y^j \frac{dy}{y^{m+2}} \right) \\ &= -\frac{1}{m^2} \sum_{i=0}^d \sum_{j=0}^{\infty} (-1)^i e_i(\tilde{\lambda}) h_j(\tilde{\lambda}) b_{ij}, \end{aligned}$$

where

$$b_{ij} = \operatorname{Res}_{y=0} \left((1 + (d - \frac{1}{2})y)(1 + (d - \frac{3}{2})y) \cdots (1 + (d - m + \frac{1}{2})y) (1 - my)^{d-i} \frac{dy}{y^{m+2-i-j}} \right).$$

Lemma 5 (i) $b_{ij} = 0$ for $i + j \geq m + 2$.

(ii) For $i + j = m + 1$, we have $b_{ij} = 1$ and

$$\sum_{i+j=m+1} (-1)^i e_i(\tilde{\lambda}) h_j(\tilde{\lambda}) b_{ij} = 0.$$

(iii) For $i + j = m$, we have $b_{ij} = \frac{m^2}{2} + mi$.

(iv) For $0 \leq i + j \leq m$, the value b_{ij} depends on d polynomially. In fact, b_{ij} is a polynomial in d of degree $m + 1 - i - j$ with coefficients in \mathbf{Q} .

Proof: (i) If $i + j \geq m + 2$, then the function inside the summation is a polynomial in y , and thus it has no pole at $y = 0$ and its residue b_{ij} is 0.

(ii) If $i + j = m + 1$, then the residue b_{ij} is 1, and the contribution to this sum is, as in I.2.6' of [8],

$$\sum_{i+j=m+1} (-1)^i e_i(\tilde{\lambda}) h_j(\tilde{\lambda}) = 0,$$

which is the coefficient of y^{m+1} in $\prod_{i=1}^d (1 - \tilde{\lambda}_i y) / \prod_{i=1}^d (1 - \tilde{\lambda}_i y) = 1$.

(iv) Since b_{ij} is the coefficient of $y^{m+1-i-j}$ in the polynomial

$$\begin{aligned} & (1 + (d - \frac{1}{2})y)(1 + (d - \frac{3}{2})y) \cdots (1 + (d - m + \frac{1}{2})y)(1 - my)^{d-i} \\ &= \sum_{s=0}^{\infty} \sum_{t=0}^m e_t(d - \frac{1}{2}, d - \frac{3}{2}, \dots, d - m + \frac{1}{2}) \binom{d-i}{s} (-m)^s y^{s+t}, \end{aligned}$$

we have

$$b_{ij} = \sum_{s=0}^{m+1-i-j} e_{m+1-i-j-s}(d - \frac{1}{2}, d - \frac{3}{2}, \dots, d - m + \frac{1}{2}) \binom{d-i}{s} (-m)^s.$$

Then b_{ij} is a polynomial in d of degree no greater than $m + 1 - i - j$.

(iii) If $i + j = m$, then

$$b_{ij} = (d - \frac{1}{2}) + (d - \frac{3}{2}) + \cdots + (d - m + \frac{1}{2}) - m(d - i) = \frac{m^2}{2} + mi.$$

This implies that

$$-\frac{1}{m^2} \sum_{i+j=m} (-1)^i e_i(\tilde{\lambda}) h_j(\tilde{\lambda}) b_{ij} = -\frac{1}{m} \sum i (-1)^i e_i(\tilde{\lambda}) h_{m-i}(\tilde{\lambda}) = \frac{1}{m} \left(\tilde{p}_m(\lambda) + \sum_{i=1}^d (-i + \frac{1}{2})^m \right).$$

□

We now return to the proof of Proposition 4. We have the finite sum expression

$$f_{\lambda}(c^{(m)}) = -\frac{1}{m^2} \sum_{i+j \leq m} (-1)^i e_i(\tilde{\lambda}) h_j(\tilde{\lambda}) b_{ij}.$$

This is a polynomial in e_i , h_j and d . We know that e_i and f_j are polynomials in power-sum symmetric functions $\tilde{p}_k(\lambda)$ and d . Then, since $d = \tilde{p}_1(\lambda)$, we have proved the existence of the function $\phi = \phi_m$. \square

Example 6 For $m = 2, \dots, 5$, the polynomial ϕ_m is of the following form:

$$\phi_2 = \frac{1}{2}Y_2, \quad \phi_3 = \frac{1}{3}Y_3 - \frac{1}{2}Y_1^2 + \frac{5}{12}Y_1, \quad \phi_4 = \frac{1}{4}Y_4 - Y_1Y_2 + \frac{11}{8}Y_2,$$

$$\phi_5 = \frac{1}{5}Y_5 - Y_3Y_2 + \frac{19}{6}Y_3 - \frac{1}{2}Y_2^2 + \frac{5}{6}Y_1^3 - \frac{15}{4}Y_1^2 + \frac{189}{80}Y_1.$$

This suggests that the degree of the polynomial ϕ_m would be m if we consider the degree of Y_j to be j . The highest order term of ϕ_m would then be $\frac{1}{m}Y_m$.

For $m = 2$, $\tilde{p}_2(\lambda)/2 = f_\lambda(c^{(2)})$ has a simple expression in terms of partitions. For a partition λ , we define $n(\lambda) = \sum_{i \geq 1} (i-1)\lambda_i$. We also define the content $c(x)$ as $c(x) = j - i$ for each box $x = (i, j) \in \lambda$. Then

$$\tilde{p}_2(\lambda)/2 = f_\lambda(c^{(2)}) = n(\lambda') - n(\lambda) = \sum_{x \in \lambda} c(x).$$

3 Quasi-modular form

3.1 Eisenstein series

We give a brief summary of quasi-modular forms to fix the notation used here. (For the precise definition and further properties, see [7] and §3 of [2].) Let τ be a complex number with $\Im \tau > 0$ and $q = e^{2\pi\sqrt{-1}\tau}$. We denote the differential operator $\frac{1}{2\pi\sqrt{-1}} \frac{d}{d\tau} = q \frac{d}{dq}$ by D . For a subgroup Γ of the full modular group $SL(2, \mathbf{Z})$ of finite index, we denote the set of modular forms of weight k by $M_k(\Gamma)$ and the graded ring of modular forms by $M_*(\Gamma) = \bigoplus_{k \geq 0} M_k(\Gamma)$. Similarly, we denote the set of quasi-modular forms of weight k by $\text{QM}_k(\Gamma)$ and the graded ring of quasi-modular forms by $\text{QM}_*(\Gamma) = \bigoplus_{k \geq 0} \text{QM}_k(\Gamma)$. The ring $M_*(\Gamma)$ is not closed under the differentiation D , but the ring $\text{QM}_*(\Gamma)$ is closed under D . Examples of (quasi-)modular forms are provided by the Eisenstein series.

We denote the Bernoulli number by $B_k \in \mathbf{Q}$, which is defined by $\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}$. For example, $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$ and $B_6 = \frac{1}{42}$.

We define the (normalized) Eisenstein series E_k for even $k \geq 4$ by

$$\begin{aligned} E_k(\tau) &= \frac{1}{2} \sum_{(c,d)=1} \frac{1}{(c\tau + d)^k} \\ &= 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{k-1} \right) q^n = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \frac{n^{k-1} q^n}{1 - q^n}. \end{aligned}$$

(This is a convergent series in q .) Then E_k is a modular form of weight k for $SL(2, \mathbf{Z})$:

$$E_k\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k E_k(\tau).$$

We also define

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \left(\sum_{d|n} d \right) q^n.$$

Then E_2 is not a modular form, but a quasi-modular form of weight 2 for $SL(2, \mathbf{Z})$, so that

$$E_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 E_2(\tau) + \frac{12}{2\pi\sqrt{-1}} c(c\tau + d).$$

Lemma 7 (i) $\text{QM}_*(\Gamma)$ is a graded \mathbf{C} -algebra, and $M_*(\Gamma) \subset \text{QM}_*(\Gamma)$ is a subalgebra.

(ii) As a \mathbf{C} -algebra, we have the isomorphism $\text{QM}_*(\Gamma) = M_*(\Gamma) \otimes \mathbf{C}[E_2]$.

(iii) $\text{QM}_*(\Gamma)$ is stable under the action of D . (It increases the degree by 2.)

For the full modular group, since $M_*(SL(2, \mathbf{Z})) = \mathbf{C}[E_4, E_6]$, we have $\text{QM}_*(SL(2, \mathbf{Z})) = \mathbf{C}[E_2, E_4, E_6]$. The differentiation D provides the dynamical system

$$D(E_2) = (E_2^2 - E_4)/12, \quad D(E_4) = (E_2 E_4 - E_6)/3, \quad D(E_6) = (E_2 E_6 - E_4^2)/2.$$

The following lemma is used for the proof of the main theorem.

Lemma 8 If $\eta(q)A(q) \in \text{QM}_k(SL(2, \mathbf{Z}))$, then $\eta(q)D^j(A(q)) \in \text{QM}_{k+2j}(SL(2, \mathbf{Z}))$ for a positive integer j .

Proof: Recall the definition of the Ramanujan delta, $\Delta(\tau) = \eta(q)^{24} = (E_4^3 - E_6^2)/1728$. Then we have $D \log \Delta(\tau) = E_2(\tau)$ and $D(\log \eta) = E_2/24$, and we obtain the formula

$$\eta(q)DA(q) = D(\eta(q)A(q)) - \frac{1}{24}E_2\eta(q)A(q).$$

The condition $\eta(q)A(q) \in \text{QM}_k(SL(2, \mathbf{Z}))$ implies $\eta(q)D(A(q)) \in \text{QM}_{k+2}(SL(2, \mathbf{Z}))$. The assertion follows from induction on j . □

3.2 The character of the infinite wedge representation

We introduce the variables t_1, t_2, t_3, \dots , where we write $D_k = \frac{\partial}{\partial t_k}$ for $k \geq 1$. In what follows, the variable t_1 is related to q by $q = e^{t_1}$. In particular, for $k = 1$ we have $D = D_1 = q \frac{\partial}{\partial q}$. We define the infinite series

$$V'(q, t_2, t_3, \dots) = \sum_{d \geq 0} \sum_{\lambda \in \mathcal{P}_d} \exp(\tilde{p}_1(\lambda)t_1 + \tilde{p}_2(\lambda)t_2 + \tilde{p}_3(\lambda)t_3 + \dots) \quad (3)$$

$$= \sum_{d \geq 0} \sum_{\lambda \in \mathcal{P}_d} q^{\tilde{p}_1(\lambda)} \exp(\tilde{p}_2(\lambda)t_2 + \tilde{p}_3(\lambda)t_3 + \dots). \quad (4)$$

This expression appears in (0.10) of [2] as a character of the infinite wedge representation of an infinite dimensional Lie algebra (W_∞) , and it is known to be a quasimodular form of weight $-\frac{1}{2}$. Let us explain this in more detail.

It is easy to see that V' is the coefficient of z^0 of an infinite product:

$$\begin{aligned} V' &= \text{Res}_{z=0} \left(\prod_{p \in \frac{1}{2} + \mathbf{Z}_{\geq 0}} (1 + z \exp(\sum_{k \geq 1} p^k t_k)) (1 + z^{-1} \exp(-\sum_{k \geq 1} (-p)^k t_k)) \frac{dz}{z} \right) \\ &= \text{Res}_{z=0} \left(\prod_{p \in \frac{1}{2} + \mathbf{Z}_{\geq 0}} (1 + z q^p \exp(\sum_{k \geq 2} p^k t_k)) (1 + z^{-1} q^p \exp(-\sum_{k \geq 2} (-p)^k t_k)) \frac{dz}{z} \right). \end{aligned}$$

To obtain a quasimodular form, we have to multiply a fractional power in t_i . Let $\xi(s) = \sum_{n \geq 1} (n - \frac{1}{2})^{-s} = (2^s - 1)\zeta(s)$, which is continued to a meromorphic function of s . The function $\xi(s)$ at negative integer values of s is well-defined, and $\xi(-2i) = 0$ for $i \in \mathbf{Z}_{>0}$. (For example, $\xi(-1) = 1/24$, $\xi(-3) = -7/960$.) We define

$$V(q, t_2, \dots) = \exp(-\sum_{j=1}^{\infty} \xi(-j)t_j) \times V'(q, t_2, \dots). \quad (5)$$

If we consider the case $t_2 = t_3 = \dots = 0$, then the infinite product reduces to

$$q^{-\xi(-1)} \prod_{p \in \frac{1}{2} + \mathbf{Z}_{\geq 0}} (1 + z q^p)(1 + z^{-1} q^p) = \frac{\sum_{n \in \mathbf{Z}} z^n q^{n^2/2}}{\eta(q)}$$

since $\xi(-1) = 1/24$. Then

$$\eta(q)V(q, 0, 0, \dots) = 1. \quad (6)$$

Now, consider the Taylor expansion of V with respect to (t_2, t_3, \dots)

$$V(q, t_2, t_3, \dots) = \sum_K A_K(q) \frac{t^K}{K!}, \quad (7)$$

where $K = (k_2, k_3, \dots)$ with almost all $k_i = 0$, and $t^K/K! = t_2^{k_2} t_3^{k_3} \dots / k_2! k_3! \dots$ is multi-index notation. The relation (6) implies that $\eta(q)A_{(0,0,\dots)}(q) = 1$. It is known in (4.8) of [2] that $\eta(q)A_K(q) \in \text{QM}_*(SL(2, \mathbf{Z}))$ and is of weight $3k_2 + 4k_3 + \dots = \sum_{i=2}^{\infty} (i+1)k_i$. By Lemma 8, we know that $\eta(q)D^j(A_K(q)) \in \text{QM}_*(SL(2, \mathbf{Z}))$ and its weight is $2j + \sum_{i=2}^{\infty} (i+1)k_i$.

3.3 Main theorem

We arrive at the stage to state our main theorem.

Theorem 9 *The counting functions $F_g(q) = F_g^{(m)}(q)$ for $g \geq 2$ belong to the graded ring $\text{QM}_*(SL(2, \mathbf{Z}))$ of quasimodular forms with respect to the full modular group $SL(2, \mathbf{Z})$. In particular, $F_g^{(m)} \in \mathbf{Q}[E_2, E_4, E_6]$.*

Proof: Summarizing Lemmas 1, 2 and 3, we obtain

$$\hat{Z}(q, X) = 1 + \sum_{b \geq 0} \sum_{d \geq 1} \sum_{\lambda \in \mathcal{P}_d} \frac{1}{b!} f_\lambda(c^{(m)})^b q^d X^b = 1 + \sum_{d \geq 1} \sum_{\lambda \in \mathcal{P}_d} \exp(f_\lambda(c^{(m)})X) q^d. \quad (8)$$

We can consider the term 1 as coming from the case $d = 0$, where $R_0 = \{\emptyset\}$, $f_\emptyset = 0$. From Proposition 4, we obtain

$$\begin{aligned} & \exp(f_\lambda(c^{(m)})X) q^d \\ &= [\exp(\phi_m(\tilde{p}_1(\lambda), \tilde{p}_2(\lambda), \dots, \tilde{p}_m(\lambda))X) \exp(t_1 \tilde{p}_1(\lambda) + t_2 \tilde{p}_2(\lambda) + \dots + t_m \tilde{p}_m(\lambda))]_{e^{t_1=q, t_2=t_3=\dots=0}} \\ &= [\exp(\phi_m(D, D_2, \dots, D_m)X) \exp(t_1 \tilde{p}_1(\lambda) + t_2 \tilde{p}_2(\lambda) + \dots + t_m \tilde{p}_m(\lambda))]_{e^{t_1=q, t_2=\dots=t_m=0}}. \end{aligned} \quad (9)$$

Then by (8), (9) and (3), we have

$$\begin{aligned} & \hat{Z}(q, X) \\ &= \left[\exp(\phi_m(D, D_2, \dots, D_m)X) \sum_{d \geq 0} \sum_{\lambda \in \mathcal{P}_d} \exp(t_1 \tilde{p}_1(\lambda) + t_2 \tilde{p}_2(\lambda) + t_3 \tilde{p}_3(\lambda) + \dots) \right]_{e^{t_1=q, t_2=t_3=\dots=0}} \\ &= [\exp(\phi_m(D, D_2, \dots, D_m)X) V'(q, t_2, t_3, \dots)]_{t_2=t_3=\dots=0} \\ &= \left[\exp(\phi_m(D, D_2, \dots, D_m)X) \exp\left(\sum_{j=1}^{\infty} t_j \xi(-j)\right) V(q, t_2, t_3, \dots) \right]_{t_2=t_3=\dots=0} \\ &= q^{1/24} [\exp(\phi_m(D + \xi(-1), D_2 + \xi(-2), \dots, D_m + \xi(-m))X) V(q, t_2, t_3, \dots)]_{t_2=t_3=\dots=0}. \end{aligned}$$

Here we have used (5). Then,

$$\begin{aligned}
& \eta(q)Z(q, X) \\
&= \eta(q)q^{-1/24}\hat{Z}(q, X) \\
&= \eta(q) [\exp(\phi_m(D + \xi(-1), D_2 + \xi(-2), \dots, D_m + \xi(-m))X)V(q, t_2, t_3, \dots)]_{t_2=t_3=\dots=0} \\
&= \eta(q) \left[\exp(\phi_m(D + \xi(-1), D_2 + \xi(-2), \dots, D_m + \xi(-m))X) \sum_K A_K(q) \frac{t^K}{K!} \right]_{t_2=t_3=\dots=0}.
\end{aligned} \tag{10}$$

The coefficient of X^b on the right-hand side of (10) is equal to the quantity

$$\frac{1}{b!} \sum_K \eta(q) \left[\phi_m(D + \xi(-1), D_2 + \xi(-2), \dots, D_m + \xi(-m))^b A_K(q) \frac{t^K}{K!} \right]_{t_2=t_3=\dots=0}.$$

This is a finite sum and belongs to $\text{QM}_*(SL(2, \mathbf{Z}))$, by Lemma 8. Then the right-hand side of (10) is a formal power series in X with coefficients in $\text{QM}_*(SL(2, \mathbf{Z}))$. Hence by (1), we have

$$\sum_{b \geq 1} F_{1+(m-1)b/2}(q) X^b / b! = \log(\eta(q)Z(q, X)) = \sum_{l=1}^{\infty} (\eta(q)Z(q, X) - 1)^l (-1)^{l-1} / l.$$

This shows that $F_q(q) \in \text{QM}_*(SL(2, \mathbf{Z}))$. □

The special case $m = 2$ of our theorem is considered by Dijkgraaf [3].

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